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N-Fractional Calculus of Some Logarithmic Functions and Some Identities (Applications of convolutions in geometric function theory)

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N -Fractional Calculus of Some Logarithmic Functions and Some Identities

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Abstract

In this article, N-fractional calculus of the logarithmic function in title is discussed.

A theorem is presented as follows for example.

Theorem 1. Let $f = f(z) = (\sqrt{z-b} - c)^2 - d \neq 0, 1.$

We have then;

(i)

$$(\log f)_\gamma = -e^{-i\pi\gamma}(z-b)^{-\gamma}\left\{\sum_{k=0}^{\infty}\frac{\Gamma(\frac{1}{2}k+\gamma)}{\Gamma(\frac{1}{2}k+1)}S^k\right.$$

$$\left.+\sum_{k=1}^{\infty}\frac{1}{k}T^k\sum_{m=0}^{\infty}\frac{[2k]_m\Gamma(\frac{1}{2}m+k+\gamma)}{m!\Gamma(\frac{1}{2}m+k)}S^m\right\}$$

$$(|\Gamma(\frac{1}{2}k+\gamma)|, |\Gamma(\frac{1}{2}m+k+\gamma)| < \infty)$$

and

(ii)

$$(\log f)_n = (-1)^{n+1}(z-b)^{-n}\Big\{\sum_{k=0}^{\infty}[\frac{1}{2}k+1]_{n-1}S^k$$

$$+\sum_{k=1}^{\infty}\frac{1}{k}T^k\sum_{m=0}^{\infty}\frac{[2k]_m[\frac{1}{2}m+k]_n}{m!}S^m\Big\}(n\in Z^+)(n\text{-th derivatives })$$

where $S=\dfrac{c}{(z-b)^{1/2}}, T=\dfrac{d}{z-b}, |S|<1, |T|<1,$

and$[\lambda]_k (k\in Z_0^+)$; Notation of Pochhammer.

§0. Introduction (Definition of Fractional Calculus)

(I) Definition. (by K. Nishimoto)([1] Vol.1)

Let $D=\{\underset{-}{\mathrm{D}},\underset{+}{\mathrm{D}}\}, C=\{\underset{-}{\mathrm{C}},\underset{+}{\mathrm{C}}\}$

$\underset{-}{\mathrm{C}}$ be a curve along the cut joining two points z and $-\infty+i\ \mathrm{Im}(z)$,

$\underset{+}{\mathrm{C}}$ be a curve along the cut joining two points z and $\infty+i\ \mathrm{Im}(z)$,

$\underset{-}{\mathrm{D}}$ be a domain surrounded by $\underset{-}{\mathrm{C}}$, $\underset{+}{\mathrm{D}}$ be a domain surrounded by $\underset{+}{\mathrm{C}}$,

(Here D contains the points over the curve C)

Moreover, let $f=f(z)$ be a regular function in D $(z\in D)$,

$$f_\nu=(f)_\nu={}_C(f)_\nu=\frac{\Gamma(\nu+1)}{2\pi i}\int_C\frac{f(\xi)}{(\xi-z)^{\nu+1}}d\xi \quad (\nu\notin \mathbf{Z}^-), \qquad (0.1)$$

$$(f)_{-m}=\lim_{\nu\to -m}(f)_\nu \quad (m\in \mathbf{Z}^+), \qquad (0.2)$$

where $-\pi\le\arg(\xi-z)\le\pi$ for $\underset{-}{\mathrm{C}}$, $0\le\arg(\xi-z)\le 2\pi$ for $\underset{+}{\mathrm{C}}$,

$$\xi\neq z,\ z\in \mathbf{C}, \nu\in\mathbf{R},\quad \Gamma: \text{Gamma function},$$

then $(f)_\nu$ is the fractional differintegration of arbitrary order ν (derivatives of order ν for $\nu>0$, and integrals of order $-\nu$ for $\nu<0$), with respect to z, of the function f, if $|(f)_\nu|<\infty$.

(II) On the fractional calculus operator N^ν [3]

Theorem A Let fractional calculus operator (Nishimoto's operator) N^ν be

$$N^\nu=\left(\frac{\Gamma(\nu+1)}{2\pi i}\int_C\frac{d\xi}{(\xi-z)^{\nu+1}}\right) \quad (\nu\notin\mathbf{Z}^-), \quad \text{[Refer to (1.1)]} \quad (0.3)$$

with

$$N^{-m} = \lim_{\nu \longrightarrow -m} N^{\nu} \quad (m \in \mathbf{Z}^{+}), \tag{0.4}$$

and define the binary operation $\circ$ as

$$N^{\beta} \circ N^{\alpha} f = N^{\beta} N^{\alpha} f = N^{\beta}(N^{\alpha} f) \quad (\alpha, \ \beta \in \mathbf{R}), \tag{0.5}$$

then the set

$$\{N^{\nu}\} = \{N^{\mu} | \nu \in \mathbf{R}\} \tag{0.6}$$

is an Abelian product group (having continuous index ν) which has the inverse transform operator $(N^{\nu})^{-1} = N^{-\nu}$ to the fractional calculus operator N^{ν} , for the function f such that $f \in F = \{f; 0 \neq |f_{\nu}| < \infty, \nu \in \mathbf{R}\}$, where $f = f(z)$ and $z \in \mathbf{C}$. (viz. $-\infty < \nu < \infty$).
(For our convenience, we call $N^{\beta} \circ N^{\alpha}$ as product of N^{β} and N^{α})
Theorem B The "F.O.G.$\{N^{\nu}\}$" is an " Action product group which has continuous index ν " for the set of F.(F.O.G.: Fractional calculus operator group)[3]

Theorem C. Let

$$S := \{\pm N^{\nu}\} \cup \{0\} = \{N^{\nu}\} \cup \{-N^{\nu}\} \cup \{0\} (\nu \in R) \tag{0.7}$$

Then the set S is a commutative ring for the function $f \in F$, when the identity

$$N^{\alpha} + N^{\beta} = N^{\gamma} (N^{\alpha}, N^{\beta}, N^{\gamma} \in S) \tag{0.8}$$

holds. [5]

(III) Lemma 1 We have [1]

(i)

$$((z-c)^{\beta})_{\alpha} = e^{-i\pi\alpha} \frac{\Gamma(\alpha - \beta)}{\Gamma(-\beta)} (z-c)^{\beta - \alpha} \quad (|\frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)}| < \infty), \tag{0.9}$$

(ii)

$$(\log(z-c))_\alpha = -e^{-i\pi\alpha}\Gamma(\alpha)(z-c)^{-\alpha} \quad (|\Gamma(\alpha)|<\infty), \quad (0.10)$$

(iii)

$$((z-c)^{-\alpha})_{-\alpha} = -e^{i\pi\alpha}\frac{1}{\Gamma(\alpha)}\log(z-c) \quad (|\Gamma(\alpha)|<\infty), \quad (0.11)$$

(iv)

$$(u\cdot v)_\alpha = \sum_{k=0}^{\infty}\frac{\Gamma(\alpha+1)}{k!\Gamma(\alpha+1-k)}u_{\alpha-k}v_k \quad (u=u(z), v=v(z)) \quad (0.12)$$

where $z\neq c$ in (0.9), and $z-c\neq 0,1$ in (0.10) and (0.11).

§1. Preliminary

Theorem D. below for the fractional calculus of a logarithmic function is reported by K. Nishimoto (cf. J. Frac. Calc. Vol. 29, May (2006), pp. 35-44)[12].

Theorem D. We have

(i)

$$(\log((z-b)^\beta - c))_\gamma = -e^{-i\pi\gamma}\beta(z-b)^{-\gamma}\Gamma(\gamma) \times\sum_{k=0}^{\infty}\frac{\Gamma(\beta k+\gamma)}{\Gamma(\gamma)\Gamma(\beta k+1)}\left(\frac{c}{(z-b)^\beta}\right)^k \quad \left(\left|\frac{\Gamma(\beta k+\gamma)}{\Gamma(\beta k+1)}\right|<\infty\right) \quad (1.1)$$

and

(ii)

$$(\log((z-b)^\beta - c))_m = (-1)^{m+1}\beta(z-b)^{-m}\Gamma(m) \times\sum_{k=0}^{\infty}\frac{\Gamma(\beta k+m)}{\Gamma(m)\Gamma(\beta k+1)}\left(\frac{c}{(z-b)^\beta}\right)^k \quad (m\in Z^+)$$

where $(z-b)^\beta - c\neq 0,1$, and $|c/(z-b)^\beta|<1$,

and $[\lambda]_k = \lambda(\lambda+1)\cdots(\lambda+k-1) = \Gamma(\lambda+k)/\Gamma(\lambda)$ with $[\lambda]_0 = 1$,

(Notation of Pochhammer). (1.2)

And the Theorem E below is reported by K. Nishimoto already (cf. JFC Vol. 32, Nov. (2007), pp. 17-28)[13].

Theorem E. We have

(i)

$$(\log(((z-b)^\beta - c)^\alpha - d))_\gamma = -e^{-i\pi\gamma}(z-b)^{-\gamma}$$

$$\times\left[\alpha\beta\sum_{k=0}^{\infty}\frac{\Gamma(\beta k+\gamma)}{\Gamma(\beta k+1)}\left(\frac{c}{(z-b)^\beta}\right)^k\right.$$

$$\left.+\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{d}{(z-b)^{\alpha\beta}}\right)^k\sum_{m=0}^{\infty}\frac{[\alpha k]_m\Gamma(\beta m+\alpha\beta k+\gamma)}{m!\Gamma(\beta m+\alpha\beta k)}\left(\frac{c}{(z-b)^\beta}\right)^m\right]$$

$$\left(\left|\frac{\Gamma(\beta k+\gamma)}{\Gamma(\beta k+1)}\right|, \left|\frac{\Gamma(\beta m+\alpha\beta k+\gamma)}{\Gamma(\beta m+\alpha\beta k)}\right| < \infty\right) \tag{1.3}$$

and

(ii)

$$(\log(((z-b)^\beta - c)^\alpha - d))_n$$

$$= (-1)^{n+1}(z-b)^{-n}\left[\alpha\beta\sum_{k=0}^{\infty}[\beta k+1]_{n-1}\left(\frac{c}{(z-b)^\beta}\right)^k\right.$$

$$\left.+\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{d}{(z-b)^{\alpha\beta}}\right)^k\sum_{m=0}^{\infty}\frac{[\alpha k]_m[\beta m+\alpha\beta k]_n}{m!}\left(\frac{c}{(z-b)^\beta}\right)^m\right]$$

where $((z-b)^\beta - c)^\alpha - d \neq 0, 1. (n \in Z^+)$

$$\text{and}\left|\frac{d}{((z-b)^\beta-c)^\alpha}\right|, \left|\frac{c}{(z-b)^\beta}\right|, \left|\frac{d}{(z-b)^{\alpha\beta}}\right| < 1. \tag{1.4}$$

§2. N-Fractional Calculus of A Logarithmic Function

Theorem 1. Let

$$f = f(z) = (\sqrt{z-b} - c)^2 - d \neq 0, 1 \tag{2.1}$$

We have then;

(i)

$$(\log f)_\gamma = -e^{-i\pi\gamma}(z-b)^{-\gamma}\left\{\sum_{k=0}^{\infty}\frac{\Gamma(\frac{1}{2}k+\gamma)}{\Gamma(\frac{1}{2}k+1)}S^k\right.$$

$$\left.+\sum_{k=1}^{\infty}\frac{1}{k}T^k\sum_{m=0}^{\infty}\frac{[2k]_m\Gamma(\frac{1}{2}m+k+\gamma)}{m!\Gamma(\frac{1}{2}m+k)}S^m\right\}$$

$$\left(\,|\Gamma(\frac{1}{2}k+\gamma)|, |\Gamma(\frac{1}{2}m+k+\gamma)| < \infty\right) \qquad (2.2)$$

and

(ii)

$$(\log f)_n = -(-1)^n(z-b)^{-n}\left\{\sum_{k=0}^{\infty}[\frac{1}{2}k+1]_{n-1}S^k\right.$$

$$\left.+\sum_{k=1}^{\infty}\frac{1}{k}T^k\sum_{m=0}^{\infty}\frac{[2k]_m[\frac{1}{2}m+k]_n}{m!}S^m\right\}(n \in Z^+)(n\text{-th derivatives })$$

$$\text{where } S = \frac{c}{(z-b)^{1/2}}, T = \frac{d}{z-b}, |S| < 1, |T| < 1, \qquad (2.3)$$

Proof of (i). Set $\beta = 1/2$ and $\alpha = 2$ in Theorem E (i), We obtain (2.2) under the conditions stated before.

Theorem 2. Let $f = f(z)$ be (2.1). We have then

(i)

$$(\log f)_\gamma = -e^{-i\pi\gamma}\frac{\Gamma(\gamma)}{2}(z-b)^{-\gamma}$$

$$\times\sum_{k=0}^{\infty}\frac{\Gamma(\frac{1}{2}k+\gamma)}{\Gamma(\gamma)\Gamma(\frac{1}{2}k+1)}\left\{\left(\frac{c+\sqrt{d}}{(z-b)^{1/2}}\right)^k+\left(\frac{c-\sqrt{d}}{(z-b)^{1/2}}\right)^k\right\}$$

$$\left(\,|\Gamma(\frac{1}{2}k+\gamma)| < \infty\right) \qquad (2.4)$$

and

(ii)

$$(\log f)_n = -(-1)^n \frac{\Gamma(n)}{2}(z-b)^{-n}$$
$$\times \sum_{k=0}^{\infty} \frac{[\frac{1}{2}k+1]_{n-1}}{\Gamma(n)} \left\{ \left(\frac{c+\sqrt{d}}{(z-b)^{1/2}} \right)^k + \left(\frac{c-\sqrt{d}}{(z-b)^{1/2}} \right)^k \right\} (n\text{-derivatives})(n \in Z+)$$
$$\text{where} \quad \left|\frac{c+\sqrt{d}}{(z-b)^{1/2}}\right|, \left|\frac{c-\sqrt{d}}{(z-b)^{1/2}}\right| < 1. \tag{2.5}$$

Proof of (i). We have

$$\log f = \log\{\sqrt{z-b} - c - \sqrt{d}\}\{\sqrt{z-b} - c + \sqrt{d}\}$$
$$= \log((z-b)^{1/2} - (c+\sqrt{d})) + \log((z-b)^{1/2} - (c-\sqrt{d})) \tag{2.6}$$

Operate N-fractional calculus operator N^γ to the both side of (2.6), we have then

$$(\log f)_\gamma = (\log((z-b)^{1/2} - (c+\sqrt{d})))_\gamma + (\log((z-b)^{1/2} - (c-\sqrt{d})))_\gamma \tag{2.7}$$

Now we have

$$(\log((z-b)^{1/2} - (c+\sqrt{d})))_\gamma = -e^{-i\pi\gamma}\frac{1}{2}(z-b)^{-\gamma}\Gamma(\gamma)$$
$$\times \sum_{k=0}^{\infty} \frac{\Gamma(\frac{1}{2}k+\gamma)}{\Gamma(\gamma)\Gamma(\frac{1}{2}k+1)} \left(\frac{c+\sqrt{d}}{(z-b)^{1/2}} \right)^k$$
$$\left(|\Gamma(\frac{1}{2}k+\gamma)| < \infty \right) \tag{2.8}$$

and

$$(\log((z-b)^{1/2} - (c-\sqrt{d})))_\gamma = -e^{-i\pi\gamma}\frac{1}{2}(z-b)^{-\gamma}\Gamma(\gamma)$$
$$\times \sum_{k=0}^{\infty} \frac{\Gamma(\frac{1}{2}k+\gamma)}{\Gamma(\gamma)\Gamma(\frac{1}{2}k+1)} \left(\frac{c-\sqrt{d}}{(z-b)^{1/2}} \right)^k$$
$$\left(|\Gamma(\frac{1}{2}k+\gamma)| < \infty \right) \tag{2.9}$$

Therefore, we obtain (2.4) from (2.7), (2.8), and (2.9) clearly.
Proof of(ii). Set $\gamma = n$ in (2.4).

§3. Some Identities

Theorem 3. We have the identities below;

(i)
$$\sum_{k=1}^{\infty}\sum_{m=0}^{\infty}\frac{[2k]_m\Gamma(\frac{1}{2}m+k+\gamma)}{k\cdot m!\,\Gamma(\frac{1}{2}m+k)}S^m T^k$$
$$=\frac{1}{2}\sum_{k=0}^{\infty}\frac{\Gamma(\frac{1}{2}k+\gamma)}{\Gamma(\frac{1}{2}k+1)}\left\{\left(\frac{c+\sqrt{d}}{(z-b)^{1/2}}\right)^k+\left(\frac{c-\sqrt{d}}{(z-b)^{1/2}}\right)^k-2S^k\right\}, \qquad (3.1)$$
$$\left(\left|\Gamma(\tfrac{1}{2}k+\gamma)\right|,\quad \left|\Gamma(\tfrac{1}{2}m+k+\gamma)\right|<\infty\right)$$

and

(ii)
$$\sum_{k=1}^{\infty}\sum_{m=0}^{\infty}\frac{[2k]_m[\frac{1}{2}m+k]_n}{k\cdot m!}S^m T^k$$
$$=2\sum_{k=0}^{\infty}[\tfrac{1}{2}k+1]_{n-1}\left\{\left(\frac{c+\sqrt{d}}{(z-b)^{1/2}}\right)^k+\left(\frac{c-\sqrt{d}}{(z-b)^{1/2}}\right)^k-2S^k\right\}\quad (n\in Z^+) \qquad (3.2)$$

where

$$S=c/(z-b)^{1/2},\quad T=d/(z-b),\quad |S|<1,\quad |T|<1,\quad |(c\pm\sqrt{d})/(z-b)^{1/2}|<1.$$

Proof of (i). It is clear from Theorem 1 (i) and Theorem 2 (i).
Proof of (ii). Set $\gamma=n$ in (i).

Corollary 1. We have the following identities;
(i)

$$(z-b-d)^{-\gamma}=\frac{1}{2}(z-b)^{-\gamma}\sum_{k=0}^{\infty}\frac{\Gamma(\frac{1}{2}k+\gamma)}{\Gamma(\gamma)\Gamma(\frac{1}{2}k+1)}$$
$$\times\left\{\left(\frac{\sqrt{d}}{(z-b)^{1/2}}\right)^k+\left(\frac{-\sqrt{d}}{(z-b)^{1/2}}\right)^k\right\}$$
$$\left(|\Gamma(\frac{1}{2}k+\gamma)|<\infty\right) \qquad (3.3)$$

and
(ii)

$$(z-b-d)^{-n} = \frac{1}{2}(z-b)^{-n}\sum_{k=0}^{\infty}\frac{[\frac{1}{2}k+1]_{n-1}}{\Gamma(n)}\left\{\left(\frac{\sqrt{d}}{(z-b)^{1/2}}\right)^k + \left(\frac{-\sqrt{d}}{(z-b)^{1/2}}\right)^k\right\}$$

$$\text{where}\left|\frac{\pm\sqrt{d}}{(z-b)^{1/2}}\right| < 1, (n \in Z^+). \tag{3.4}$$

Proof of (i). Set $c = 0$ in (1).
Indeed we have

$$(\log(z-b-d))_\gamma = -e^{-i\pi\gamma}\Gamma(\gamma)(z-b-d)^{-\gamma}(|\Gamma(\gamma)| < \infty) \tag{3.5}$$

from (2.2), setting $c = 0$.
Next we have

$$(\log(z-b-d))_\gamma = -e^{-i\pi\gamma}\frac{\Gamma(\gamma)}{2}(z-b)^{-\gamma}$$
$$\times\sum_{k=0}^{\infty}\frac{\Gamma(\frac{1}{2}k+\gamma)}{\Gamma(\gamma)\Gamma(\frac{1}{2}k+1)}\left\{\left(\frac{\sqrt{d}}{(z-b)^{1/2}}\right)^k + \left(\frac{-\sqrt{d}}{(z-b)^{1/2}}\right)^k\right\}$$
$$(|\Gamma(\frac{1}{2}k+\gamma)| < \infty) \tag{3.6}$$

from (2.4), setting $c = 0$.
Therefore we have

$$(z-b-d)^{-\gamma} = \frac{1}{2}(z-b)^{-\gamma}\sum_{k=0}^{\infty}\frac{\Gamma(\frac{1}{2}k+\gamma)}{\Gamma(\gamma)\Gamma(\frac{1}{2}k+1)}$$
$$\times\left\{\left(\frac{\sqrt{d}}{(z-b)^{1/2}}\right)^k + \left(\frac{-\sqrt{d}}{(z-b)^{1/2}}\right)^k\right\} \tag{3.7}$$

from (3.5) and (3.6).

Proof of (ii). Set $\gamma = n$ in (3.3).

§4. Semi Derivatives

I We have

$$(\log((\sqrt{z-b}-c)^2-d))_{1/2}=i(z-b)^{-1/2}$$
$$\times\left[\sum_{k=0}^{\infty}\frac{\Gamma(\frac{1}{2}k+\frac{1}{2})}{\Gamma(\frac{1}{2}k+1)}S^k+\sum_{k=1}^{\infty}\frac{1}{k}T^k\sum_{m=0}^{\infty}\frac{[2k]_m\Gamma(\frac{1}{2}m+k+\frac{1}{2})}{m!\Gamma(\frac{1}{2}m+k)}S^m\right] \tag{4.1}$$

(S is the one shown in Theorem 1) (semi derivatives) from Theorem 1.(i), setting $\gamma=1/2$.

II We have

$$(\log((\sqrt{z-b}-c)^2-d))_{1/2}=i\frac{\Gamma(\frac{1}{2})}{2}(z-b)^{-1/2}$$
$$\times\sum_{k=0}^{\infty}\frac{\Gamma(\frac{1}{2}k+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}k+1)}\left\{\left(\frac{c+\sqrt{d}}{(z-b)^{1/2}}\right)^k+\left(\frac{c-\sqrt{d}}{(z-b)^{1/2}}\right)^k\right\} \tag{4.2}$$

from Theorem 2. (i).

III

$$(\log(\sqrt{z-b}-c)^2)_{1/2}=i(z-b)^{-1/2}\sum_{k=0}^{\infty}\frac{\Gamma(\frac{1}{2}k+\frac{1}{2})}{\Gamma(\frac{1}{2}k+1)}S^k. \tag{4.3}$$

(semi derivatives)

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